A Study for Obtaining more Compacton Solutions of the Modified Form of Fifth-order Korteweg-De Vries-like Equations

Mustafa Inc

Department of Mathematics, Firat University, Elazig 23119, Turkey

Reprint requests to Dr. M. I.; E-mail:minc@firat.edu.tr

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In this paper we investigate exact solutions of a modified form of fifth-order Korteweg-de Vries-like equations by using two direct methods. Thus we get new compacton solutions having infinite wings or tails. In addition, new periodic and singular periodic wave solutions are obtained.

Key words: Modified Form of Fifth-Order KdV-like Equations; Compacton; Periodic Wave Solution.

1. Introduction

In the beginning of the 1990's, Rosenau and Hyman [1] introduced the $K(m,n)$ partial differential equations (PDEs)

$$u_t + a\,(u^m)_x + (u^n)_{xxx} = 0, m > 1, 1 \leq n \leq 3, \quad (1)$$

which are a generalization of the Korteweg-de Vries (KdV) equation. For certain values of m and n, the $K(m,n)$ equation yields solitary waves. For $m = n$ these solitary waves, take the particularly simple form

$$u = \left\{ \frac{2n\lambda}{n+1} \cos\left[\frac{n-1}{4n}(x-\lambda t) \right] \right\}^{2/n-1}, \quad (2)$$

where $|x-\lambda t| \leq \dfrac{2n\pi}{n-1}$, and $u = 0$ otherwise.

For $a < 0$ one obtains solitary patterns having cusps or infinite slopes [2]. They discovered solitary waves, called compactons, with a compact support characterized by the absence of infinite wings or the absence of infinite tails. If $a = 1$, then (1) has a focusing $(+)$ branch that exhibits compacton solutions. If $a = -1$, then (1) has a defocusing $(-)$ branch that exhibits solitary pattern solutions. More studies of compacton and solitary pattern solutions of the nonlinear dispersive equations can be found in [1 – 13].

We consist the following modified nonlinear dispersive fifth-order KdV-like equations (shortly called $mfK(m,n,k,p)$ for (3)) in higher-dimensional spaces:

$$u^{m-1}u_t + a(u^n)_x + b(u^k)_{xxx} + (u^p)_{xxxxx} = 0, \quad (3)$$

$$u^{m-1}u_t + a(u^n)_x + b(u^k)_{xxx} + (u^p)_{xxxxx} + (u^q)_{yyyyy} = 0, \quad (4)$$

$$u^{m-1}u_t + a(u^n)_x + b(u^k)_{xxx} + (u^p)_{xxxxx} + (u^q)_{yyyyy} + (u^r)_{zzzzz} = 0, \quad (5)$$

where a and b are real and m,n,p,q,k and r are integers. If we put the coefficient of the term with $(u^p)_{xxxxx}$ to zero, then we get the modified nonlinearly dispersive $mK(m,n,k)$ equations. Compacton and solitary pattern solutions of this equation were investigated by Yan [3].

A compacton is a soliton solution which has a finite wavelength or is free of exponential wings. Unlike the soliton, that narrows as the amplitude increases, the width of the compacton is independent of its amplitude. Compacton solutions have been used in many scientific applications such as the super deformed nuclei, phonon, photon, the fission of liquid drops (nuclear physics), pre-formation of clusters in hydrodynamic models, inertial fusion as also indicated by Wazwaz [5 – 7].

Many mathematical methods have been involved in the compacton concept, such as the Bäcklund transformation, the Painlevé analysis, the inverse scattering method and the Darboux transformation. Compacton solutions have also been standards by many numerical methods, for example the finite difference method [8], the pseudo-spectral method [9] and the Adomian decomposition method [10].

The purpose of this work is to obtain compacton solutions of (3) easier than with the above methods, which require more time and more calculations. In addition, we obtain new compacton and solitary pattern solutions of (4) and (5).

The rest of this paper is organized as follows: In Sect. 2 we give some compacton solutions and some periodic and singular periodic wave solutions of (3). In the last Section some conclusions are drawn.

2. General Formulas of Compacton Solutions of Equation (3)

We first make the travelling wave transformation as follows:

$$u(x,y,t) = U(\xi),\ \xi = x - \lambda t, \tag{6}$$

where λ is a constant. Then (3) reduces to

$$-\lambda U^{m-1}\frac{\mathrm{d}U}{\mathrm{d}\xi} + a\frac{\mathrm{d}U^n}{\mathrm{d}\xi} + b\frac{\mathrm{d}^3U^k}{\mathrm{d}\xi^3} + \frac{\mathrm{d}^5U^p}{\mathrm{d}\xi^5} = 0. \tag{7}$$

Integrating (7) once and setting the integration constant to zero, we have

$$-\frac{\lambda}{m}U^m + aU^n + b\left(U^k\right)'' + (U^p)'''' = 0. \tag{8}$$

To seek compacton solutions of (3), we assume that (8) has solutions in the forms given in [11–13].

Type A:

$$u(x,y,t) = U(\xi) = \begin{cases} P\cos^{\beta}(R\xi), & |R\xi| \le \dfrac{\pi}{2}, \\ 0, & |R\xi| > \dfrac{\pi}{2}. \end{cases} \tag{9}$$

Type B:

$$u(x,y,t) = U(\xi) = \begin{cases} P\sin^{\beta}(R\xi), 0 \le R\xi \le \pi, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{10}$$

where P, R and β are constants to be determined later.

Substituting (9) into (8), gives

$$\begin{aligned} &-\frac{\lambda}{m}P^m\cos^{m\beta}(R\xi) + aP^n\cos^{n\beta}(R\xi) \\ &+ bk\beta(k\beta-1)R^2P^k\cos^{k\beta-2}(R\xi) \\ &- bk^2\beta^2P^k\cos^{k\beta}(R\xi) \\ &+ p\beta(11p\beta - 6p^2\beta^2 - 6)P^pR^4\cos^{p\beta-4}(R\xi) \\ &+ p\beta(p^3\beta^3 - 6p^2\beta^2 + 14p\beta - 8)R^4P^p\cos^{p\beta-2}(R\xi) \\ &+ p\beta(3p\beta-2)p\beta P^pR^4\cos^{p\beta}(R\xi) = 0. \end{aligned} \tag{11}$$

Thus, with the aid of Mathematica, from (11) we easily have

$$\begin{aligned} &p = n, \quad \beta = \frac{4}{p-m}, \\ &P^{p-m} = -\frac{\lambda(5p+m)(p-m)}{3m^2 - 29p^2 - 16pm}, \\ &R = \left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}, \end{aligned} \tag{12}$$

where $3p\beta - 2 \ne 0$.

$$\begin{aligned} &k = m, \quad \beta = \frac{4}{p-n}, \\ &P^{p-n} = -\frac{\lambda^2(p-n)(29p^2 + 16pn + 3n^2)}{32k^4b^2m^2}, \\ &R = \sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}, \end{aligned} \tag{13}$$

where $11p\beta - 6p^2\beta^2 - 6 \ne 0$.

$$\begin{aligned} &p = k, \quad \beta = -\frac{2}{k-m}, \\ &P^{k-m} = \frac{\lambda(k-m)^2(3p-k+m)}{2mbk(k+m)}, \\ &R = k\sqrt{\frac{b}{3p-k+m}}, \end{aligned} \tag{14}$$

where $k\beta - 1 \ne 0$.

$$\begin{aligned} &n = k, \quad \beta = \frac{2}{k-m}, \quad P^{k-m} = \frac{2\lambda k}{am(m+k)}, \\ &R = \sqrt{\frac{a}{b}}\frac{k-m}{2k}, \end{aligned} \tag{15}$$

where $k\beta - 1 \ne 0$.

$$\begin{aligned} &k = n, \quad \beta = \frac{2}{p-k}, \\ &P^{p-k} = -\frac{bk(3k-p)(p-k)}{2p(4p^2 + 5kp + 3k^2)}, \\ &R = \sqrt{\frac{a}{b}}\frac{p-k}{2k}, \end{aligned} \tag{16}$$

where $k\beta - 1 \ne 0$ and $11p\beta - 6p^2\beta^2 - 6 \ne 0$.

$$\begin{aligned} &k = m, \quad \beta = \frac{2}{k-n}, \quad P^{n-k} = \frac{\lambda(k+n)}{2akm}, \\ &R = \sqrt{-\frac{\lambda}{bm}}\frac{k-n}{2k}, \end{aligned} \tag{17}$$

where $k\beta - 1 \neq 0$.

$$m = n = k, \quad \beta = \frac{1}{k}, \\ R = \sqrt{\frac{ma-\lambda}{m}}, \quad k\beta - 1 = 0. \tag{18}$$

$$m = k = p, \quad \beta = \frac{2}{3p}, \\ R = \sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}, \quad 3p\beta - 2 = 0. \tag{19}$$

$$n = k = p, \quad \beta = \frac{2}{3p}, \\ R = \sqrt{\frac{a}{b}\frac{3p}{2k}}, \quad 3p\beta - 2 = 0. \tag{20}$$

$$m = n = k = p, \quad \beta = \frac{2}{3p}, \\ R = \sqrt{\frac{ma-\lambda}{bm}}, \quad 3p\beta - 2 = 0. \tag{21}$$

Therefore from (9) and (12)–(21), we have the following conclusions:

2.1. *Compacton and Periodic Wave Solutions of Equation (3)*

Type A: Substituting (9) into (8), we can obtain the following compacton and periodic wave solutions of (3):

Case I. When $p = n \neq m, k$, then the exact solution of the $mfK(m,n,k,n)$ equations are

$$u = \left\{-\frac{\lambda(5p+m)(p-m)}{3m^2 - 29p^2 - 16pm} \cdot \cos^4\left[\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right]\right\}^{\frac{1}{p-m}}, \tag{22}$$

where $\left|\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right| > \pi/2$.

When $p = n < m, k$, we get periodic wave solutions:

$$u = \left\{-\frac{3m^2 - 29p^2 - 16pm}{\lambda(5p+m)(p-m)} \cdot \sec^4\left[\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right]\right\}^{\frac{1}{m-p}}, \tag{23}$$

where $\left|\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right| \geq \pi/2$.

Case II. When $m = k \neq n, p$, the exact solution of the $mfK(m,n,m,p)$ equations are given by

$$u = \left\{-\frac{\lambda^2(p-n)(29p^2 + 16pn + 3n^2)}{32k^4b^2m^2} \cdot \cos^4\left[\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right]\right\}^{\frac{1}{p-n}}, \tag{24}$$

where $\left|\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right| > \pi/2$.

$$u = \left\{\frac{\lambda(k+n)}{2akm}\cos^2\left[\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right]\right\}^{\frac{1}{n-k}}, \tag{25}$$

where $\left|\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right| > \pi/2$.

Remark 1. We know that (25) is a soliton solution with a compact support which is similar to the compacton solution of the $K(n,n)$ equation by Wazwaz [5].

While, when $m = k < n, p$, it is easy to see that (24) and (25) become periodic wave solutions

$$u = \left\{-\frac{32k^4b^2m^2}{\lambda^2(p-n)(29p^2 + 16pn + 3n^2)} \cdot \sec^4\left[\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right]\right\}^{\frac{1}{n-p}}, \tag{26}$$

where $\left|\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right| \geq \pi/2$,

$$u = \left\{\frac{2akm}{\lambda(k+n)}\sec^2\left[\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right]\right\}^{\frac{1}{k-n}}, \tag{27}$$

where $\left|\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right| \geq \pi/2$.

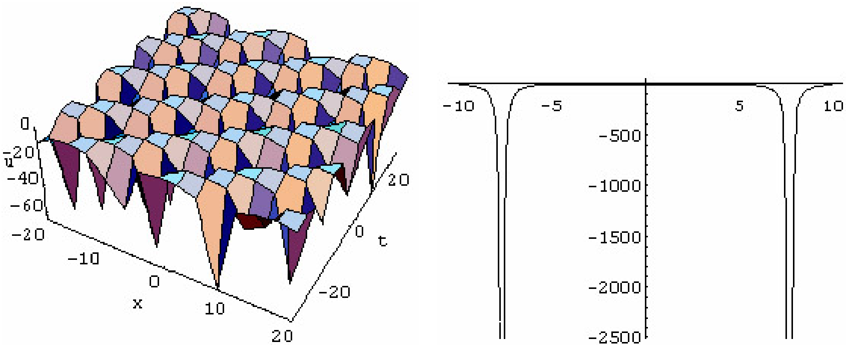

Fig. 1. The surface shows the singular periodic wave solution of (27) for $a = b = -1$, $n = \lambda = 1$, $m = 3$, and $k = 2$, and its plot at $t = 0$.

Case III. When $p = k \neq m, n$, the compacton solution of the $mfK(m,n,k,k)$ equations is given by

$$u = \left\{ \frac{\lambda(k-m)^2(3p-k+m)}{2mbk(k+m)} \cdot \cos^2\left[k\sqrt{\frac{b}{3p-k+m}}(\xi)\right]\right\}^{\frac{1}{m-k}}, \tag{28}$$

where $\left|k\sqrt{\frac{b}{3p-k+m}}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|k\sqrt{\frac{b}{3p-k+m}}(\xi)\right| > \pi/2$, while, when $p = k < n, p$, it is easy to see that (28) becomes the periodic wave solution

$$u = \left\{ \frac{2mbk(k+m)}{\lambda(k-m)^2(3p-k+m)} \cdot \sec^2\left[k\sqrt{\frac{b}{3p-k+m}}(\xi)\right]\right\}^{\frac{1}{k-m}}, \tag{29}$$

where $\left|k\sqrt{\frac{b}{3p-k+m}}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|k\sqrt{\frac{b}{3p-k+m}}(\xi)\right| \geq \pi/2$.

Case IV. For $n = k \neq m, p$, the exact solution of the $mfK(m,n,n,p)$ equations are given by

$$u = \left\{ \frac{2\lambda k}{ma(k+m)} \cos^2\left[\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right]\right\}^{\frac{1}{k-m}}, \tag{30}$$

where $\left|\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right| > \pi/2$,

$$u = \left\{ -\frac{bk(3k-p)(p-k)}{2p(4p^2+5kp+3k^2)} \cdot \cos^2\left[\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right]\right\}^{1/(p-k)}, \tag{31}$$

where $\left|\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right| \leq \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right| > \pi/2$.

Remark 2. We know that (30) is a compact solution which equals the $mK(m,n,k)$ equation (38) by Yan [14].

While when $n = k < m, p$, it is easy to see that (30) and (31) become periodic wave solutions

$$u = \left\{ \frac{ma(k+m)}{2\lambda k} \sec^2\left[\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right]\right\}^{\frac{1}{m-k}}, \tag{32}$$

where $\left|\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right| \geq \pi/2$,

$$u = \left\{ -\frac{2p(4p^2+5kp+3k^2)}{bk(3k-p)(p-k)} \cdot \sec^2\left[\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right]\right\}^{\frac{1}{k-p}}, \tag{33}$$

where $\left|\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right| \geq \pi/2$.

Remark 3. We know that (30) and (32) are soliton solutions with compact support and periodic wave solutions which are same the Eqs. (18) and (19), respectively, obtained in [4].

Case V. When $m = n = k$, the exact solution of the $mfK(m,m,m,p)$ equations is given by

$$u = P\left\{\cos\left[\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right]\right\}^{1/k}, \tag{34}$$

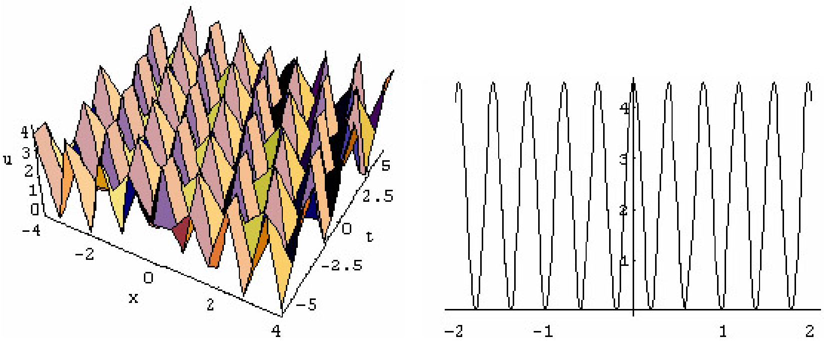

Fig. 2. The surface shows the periodic wave solution of (32) for $a = b = \lambda = 1$, $m = 3$ and $k = 4$, and its plot at $t = 0$.

where P is real or complex, $\left|\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right| \le \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right| > \pi/2$.

Remark 4. For $m = n = k$, we know that (34) is a compacton solution which is similar to Eq. (22) in [4], while when $m = n = k < p$, it is easy to see that (34) becomes

$$u = P\left\{\sec\left[\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right]\right\}^{-1/k}, \quad (35)$$

which is the singular periodic wave solution of (3) that is not a soliton solution with a compact support [1], where $\left|\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right| \ge \pi/2$.

Case VI. We can obtain when $m = k = p$, the exact solution of the $mfK(m,n,m,m)$ equations is given by

$$u = P\left\{\cos^2\left[\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right]\right\}^{1/3p}, \quad (36)$$

where $\left|\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right| \le \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right| > \pi/2$, while when $m = k = p < n$, it is easy to see that (36) becomes

$$u = P\left\{\sec^2\left[\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (37)$$

which is the singular periodic wave solution of (3), where $\left|\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right| \ge \pi/2$

Case VII. When $n = k = p \ne m$, the exact solution of the $mfK(m,n,n,n)$ equations is given by

$$u = P\left\{\cos^2\left[\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right]\right\}^{1/3p}, \quad (38)$$

where $\left|\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right| \le \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right| > \pi/2$, while when $n = k = p < m$, it is easy to see that (38) becomes

$$u = P\left\{\sec^2\left[\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (39)$$

which is the singular periodic wave solution of (3), where $\left|\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right| < \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{a}{b}\frac{3p}{2k}}(x-\lambda t)\right| \ge \pi/2$.

Case VIII. When $m = n = k = p$, the exact solution of the $mfK(m,m,m,m)$ equations is given by

$$u = P\left\{\cos^2\left[\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right]\right\}^{1/3p}, \quad (40)$$

where $\left|\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right| \le \pi/2$ and $u = 0$ for $\left|\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right| > \pi/2$.

Remark 5. For $m = n = k$, we know that (40) is a compacton solution which equal to Eq. (22) obtained in [4], while when $n = k = p < m$, it is easy to see that (40) becomes

$$u = P\left\{\sec^2\left[\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (41)$$

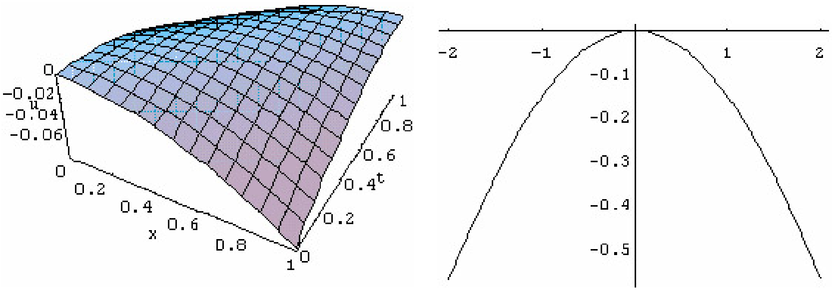

Fig. 3. The surface shows the compacton solution of (45) for $n=\lambda=1$, $m=3$, $a=b=-1$ and $k=2$, and its plot at $t=0$.

which is the singular periodic wave solution of (3), where $\left|\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right| < \pi/2$ and $u=0$ for $\left|\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right| \geq \pi/2$.

Type B: Similar to Case I, substituting (10) into (8), we can obtain other compacton and periodic wave solutions of (3).

Case I. When $p=n\neq m,k$, then the exact solution of the $mfK(m,n,k,n)$ equations are given by

$$u=\left\{-\frac{\lambda(5p+m)(p-m)}{3m^2-29p^2-16pm}\cdot\sin^4\left[\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right]\right\}^{\frac{1}{p-m}}, \tag{42}$$

where $0\leq\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(x-\lambda t)\leq\pi$ and $u=0$, otherwise. When $p=n<m,k$, we get periodic wave solutions

$$u=\left\{-\frac{3m^2-29p^2-16pm}{\lambda(5p+m)(p-m)}\cdot\csc^4\left[\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(\xi)\right]\right\}^{\frac{1}{m-p}}, \tag{43}$$

where $0<\left(-\frac{a(p-m)^2}{8p(5p+m)}\right)^{1/4}(x-\lambda t)<\pi$ and $u=0$, otherwise.

Case II. When $m=k\neq n,p$, the exact solution of the $mfK(m,n,m,p)$ equations is given by

$$u=\left\{-\frac{\lambda^2(p-n)(29p^2+16pn+3n^2)}{32k^4b^2m^2}\cdot\sin^4\left[\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right]\right\}^{\frac{1}{p-n}}, \tag{44}$$

where $0\leq\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\leq\pi$ and $u=0$, otherwise.

$$u=\left\{\frac{\lambda(k+n)}{2akm}\sin^2\left[\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right]\right\}^{\frac{1}{n-k}}, \tag{45}$$

where $0\leq\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\leq\pi$ and $u=0$, otherwise, while when $m=k<n,p$, it is easy to see that (44) and (45) become another periodic wave solutions

$$u=\left\{-\frac{32k^4b^2m^2}{\lambda^2(p-n)(29p^2+16pn+3n^2)}\cdot\csc^4\left[\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right]\right\}^{\frac{1}{n-p}}, \tag{46}$$

where $0<\left|\sqrt{-\frac{\lambda}{bm}\frac{p-n}{4k}}(\xi)\right|<\pi$ and $u=0$, otherwise.

$$u=\left\{\frac{2akm}{\lambda(k+n)}\csc^2\left[\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)\right]\right\}^{\frac{1}{k-n}}, \tag{47}$$

where $0<\sqrt{-\frac{\lambda}{bm}\frac{k-n}{2k}}(\xi)<\pi$ and $u=0$, otherwise.

Case III. When $p=k\neq m,n$, the exact solution of the $mfK(m,n,k,k)$ equations is given by

$$u=\left\{\frac{\lambda(k-m)^2(3p-k+m)}{2mbk(k+m)}\cdot\sin^2\left[k\sqrt{\frac{b}{3p-k+m}}(\xi)\right]\right\}^{\frac{1}{m-k}}, \tag{48}$$

where $0\leq k\sqrt{\frac{b}{3p-k+m}}(\xi)\leq\pi$ and $u=0$, otherwise, while when $p=k<n,p$, it is easy to see that (48)

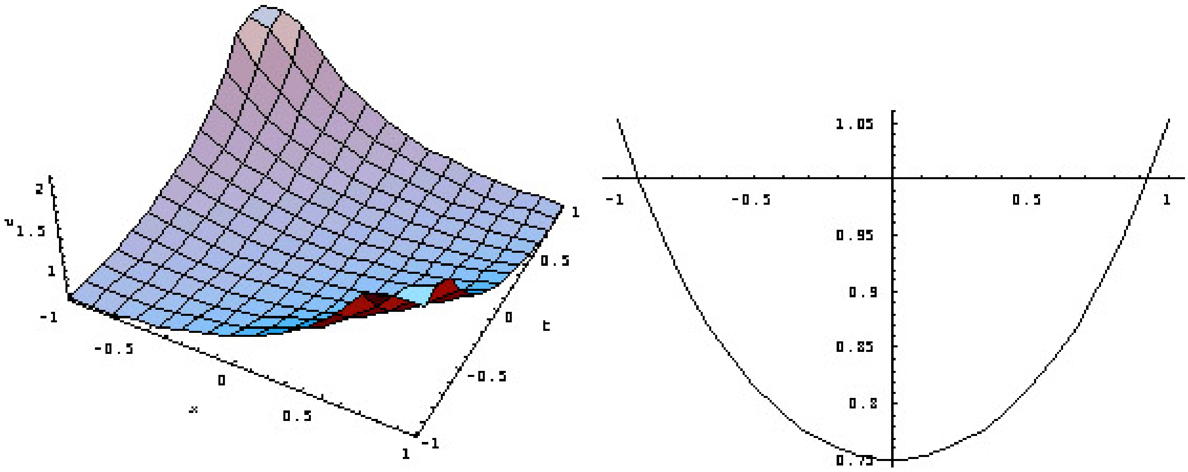

Fig. 4. The surface shows the compacton solution of (50) for $a = b = m = \lambda = 1$ and $k = 4$, and its plot at $t = 0$.

becomes a periodic wave solution

$$u = \left\{ \frac{2mbk(k+m)}{\lambda(k-m)^2(3p-k+m)} \cdot \csc^2\left[k\sqrt{\frac{b}{3p-k+m}}(\xi)\right]\right\}^{1/(k-m)}, \quad (49)$$

where $0 < k\sqrt{\frac{b}{3p-k+m}}(\xi) < \pi$ and $u = 0$, otherwise.

Case IV. When $n = k \neq m, p$, the exact solution of the $mfK(m,n,n,p)$ equations is given by

$$u = \left\{ \frac{2\lambda k}{ma(k+m)} \sin^2\left[\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right]\right\}^{\frac{1}{k-m}}, \quad (50)$$

where $0 \leq \sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi) \leq \pi$ and $u = 0$, otherwise.

$$u = \left\{ -\frac{bk(3k-p)(p-k)}{2p(4p^2+5kp+3k^2)} \cdot \sin^2\left[\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right]\right\}^{\frac{1}{p-k}}, \quad (51)$$

where $0 \leq \sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi) \leq \pi$ and $u = 0$, otherwise, while when $n = k < m, p$, it is easy to see that (50) and (51) becomes periodic wave solutions

$$u = \left\{ \frac{ma(k+m)}{2\lambda k} \csc^2\left[\sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi)\right]\right\}^{\frac{1}{m-k}}, \quad (52)$$

where $0 < \sqrt{\frac{a}{b}}\frac{k-m}{2k}(\xi) < \pi$ and $u = 0$, otherwise,

$$u = \left\{ -\frac{2p(4p^2+5kp+3k^2)}{bk(3k-p)(p-k)} \cdot \csc^2\left[\sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi)\right]\right\}^{1/(k-p)}, \quad (53)$$

where $0 < \sqrt{\frac{a}{b}}\frac{p-k}{2k}(\xi) < \pi$ and $u = 0$, otherwise.

Remark 6. We know that (50) and (52) are soliton solutions with compact support and periodic wave solutions which are equal to the Eqs. (24) and (25) obtained in [4].

Case V. When $m = n = k$, the exact solution of the $mfK(m,m,m,p)$ equations is given by

$$u = P\left\{ \sin\left[\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right]\right\}^{1/k}, \quad (54)$$

where $0 \leq \sqrt{\frac{ma-\lambda}{m}}(x-\lambda t) \leq \pi$ and $u = 0$, otherwise.

While when $m = n = k < p$, it is easy to see that (54) becomes

$$u = P\left\{ \csc\left[\sqrt{\frac{ma-\lambda}{m}}(x-\lambda t)\right]\right\}^{-1/k}, \quad (55)$$

which is another singular periodic wave solution of (3), where $0 < \sqrt{\frac{ma-\lambda}{m}}(x-\lambda t) < \pi$ and $u = 0$, otherwise.

Case VI. When $m = k = p$, the exact solution of the $mfK(m,n,m,m)$ equations is

$$u = P\left\{\sin^2\left[\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right]\right\}^{1/3p}, \quad (56)$$

where $0 \le \sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t) \le \pi$ and $u = 0$, otherwise, while when $m = k = p < n$, it is easy to see that (56) becomes

$$u = P\left\{\csc^2\left[\sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (57)$$

which is another singular periodic wave solution of (3), where $0 < \sqrt{-\frac{3\lambda p}{2bm}\frac{1}{k}}(x-\lambda t) < \pi$ and $u = 0$, otherwise.

Case VII. When $n = k = p \ne m$, the another compact solution of the $mfK(m,n,n,n)$ equation is given by

$$u = P\left\{\sin^2\left[\sqrt{\frac{a}{b}}\frac{3p}{2k}(x-\lambda t)\right]\right\}^{1/3p}, \quad (58)$$

where $0 \le \sqrt{\frac{a}{b}}\frac{3p}{2k}(x-\lambda t) \le \pi$ and $u = 0$, otherwise, while when $n = k = p < m$, it is easy to see that (58) becomes

$$u = P\left\{\csc^2\left[\sqrt{\frac{a}{b}}\frac{3p}{2k}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (59)$$

which is another singular periodic wave solution of (3), where $0 < \sqrt{\frac{a}{b}}\frac{3p}{2k}(x-\lambda t) < \pi$ and $u = 0$, otherwise.

Case VIII. When $m = n = k = p$, the exact solution of the $mfK(m,m,m,m)$ equations is given by

$$u = P\left\{\sin^2\left[\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right]\right\}^{1/3p}, \quad (60)$$

where $0 \le \sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t) \le \pi$ and $u = 0$, otherwise.

Remark 7. For $m = n = k$, we know that (60) is a compacton solution which is similar to Eq. (28) obtained in [4], while when $n = k = p < m$, it is easy to see that (60) becomes

$$u = P\left\{\csc^2\left[\sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t)\right]\right\}^{-1/3p}, \quad (61)$$

which is another singular periodic wave solution of (3), that is not a soliton solution with a compact support [1], where $0 < \sqrt{\frac{ma-\lambda}{bm}}(x-\lambda t) < \pi$ and $u = 0$, otherwise.

3. Conclusions

In this paper, we obtained a rich variety of compacton solutions of the $mfK(m,n,k,p)$ equations. We also give some periodic wave and singular periodic wave solutions of this equation. The basic goals of this study have been to extend the work of Wazwaz [5] on variants of the $fK(n,n)$, $fK(n,n,n)$ and $fK(n,n,n,n)$ equations. Two types of problems, the focusing branch and the defocusing branch were studied by Wazwaz. He has established two distinct general sets of formulas, the first has the exponent $1/n$ and the second has the exponent $2/n$. Our solutions also include the solutions obtained by Yan [4] and Wazwaz [5]. In this work, we found more compacton and periodic wave solutions of the $mfK(m,n,k,p)$ equations, and many new solutions not found before.

The present method is direct and efficient to obtain new compacton and periodic wave solutions of (3). Our method can very easily be applied to this type modified nonlinear dispersive and nonlinear dispersive equations in higher-dimensional spaces. The method has the important advantage that the equations are handled directly with a minimum of calculations.

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